SOURCES OF MAGNETIC FIELD

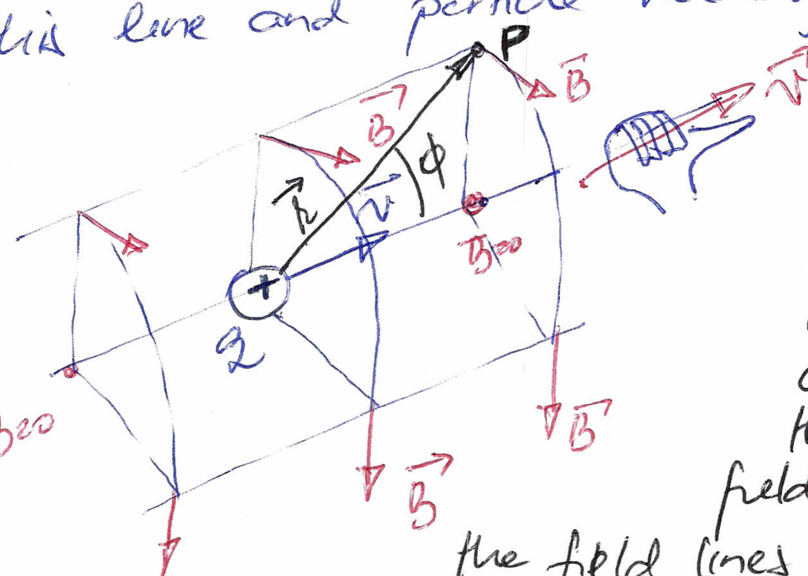
We have studied the forces exerted on moving charges and on current carrying conductors in magnetic field. The question is how the magnetic field was created?

We will show that the magnetic field is created by moving charges. We will introduce Ampere's law which plays a role in magnetism analogous to the Gauss law in electrostatics. Ampere's law let us explain symmetry properties in relating magnetic fields to their sources.

① Magnetic field of a moving charge

Let's start with the basis: the magnetic field of a single point charge q moving with a constant velocity $\vec{v}$. In practical applications, in wires, solenoids, ..., the magnetic field will be produced by a large number of charges moving together in a current.

Experiments show that $|\vec{B}|$ is proportional to $|q|$ and to $1/r^2$ (similar to $\vec{E}$). But, the direction of $\vec{B}$ is not along the line from source to point P. Instead, $\vec{B}$ is perpendicular to the plane containing this line and particle velocity $\vec{v}$



Right hand rule:
The thumb indicate the direction of velocity and the fingers curl around the charge in the direction of magnetic field lines. If charge is negative the field lines are in opposite direction

$$\boxed{B = \frac{\mu_0}{4\pi} |q| \frac{v \sin\phi}{r^2}}$$

proportionality constant.

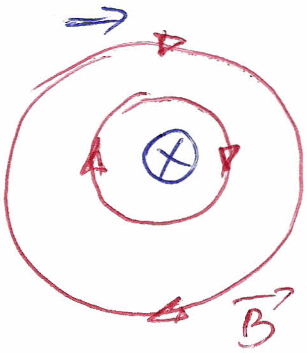
Vector magnetic field

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times \hat{r}}{r^2}}$$

$\hat{r} = \frac{\vec{r}}{r}$; $v \sin\phi = \vec{v} \times \hat{r}$

magnetic field of a point charge with constant velocity.

Magnetic field lines



$\otimes$ charge moving into the plane of the page

→ direction of $\vec{B}$ given by the right hand rule: the thumb indicates the $\vec{v}$, the fingers indicate $\vec{B}$.

In SI

$$\mu_0 = 4\pi \cdot 10^{-7}\, Ns^2/C^2 = 4\pi \cdot 10^{-7}\, Wb/Am = 4\pi \cdot 10^{-7}\, Tm/A$$

Obs: we'll see when studying electromagnetic waves that their speed of propagation in vacuum is:

$$\boxed{c^2 = \frac{1}{\varepsilon_0 \mu_0}}$$

(2) Magnetic field of a current element

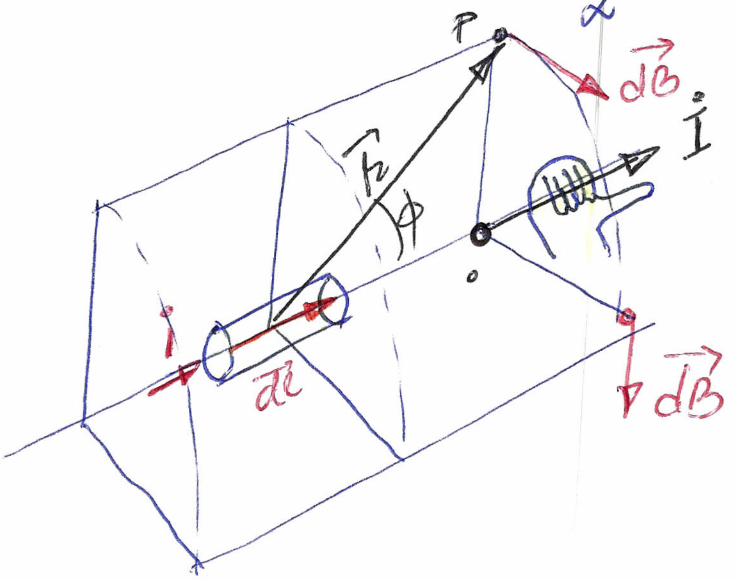
Just as for the electric field, there is a principle of superposition of magnetic fields

The total magnetic field caused by several moving charges is the vector sum of fields caused by individual charges

This principle can be used to calculate the magnetic field produced by a current into a conductor,

We consider a short element $\vec{dl}$ (fig) of volume $A\,dl$ (A = cross sectional area of conductor). If n moving charges / volume, the total moving charge is:

$$dQ = nq\,A\,dl$$



Right hand rule: indicate $\vec{B}$ direction

The moving charge in this segment are equivalent to a single charge dQ travelling with the drift velocity $\vec{v_d}$. $\Longrightarrow$

$$dB = \frac{\mu_0}{4\pi} \frac{|dQ|\,v_d \sin\phi}{r^2} = \frac{\mu_0}{4\pi} \frac{n|q|\,v_d A\,dl \sin\phi}{r^2}$$

but : $\underline{\dot{I} = n|q|\,v_d\,A}$ is the current

$$\Rightarrow \boxed{dB = \frac{\mu_0}{4\pi} \frac{\dot{I}\,dl \sin\phi}{r^2}}$$

law Biot and Savart

Vector magnetic field: current element

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^2}}$$

law of BIOT and SAVART

The total magnetic field in any point $P(\vec{r})$ in a complete circuit is calculated by integrating over all segments $d\vec{l}$ that carry current:

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l} \times \hat{r}}{r^2}} \quad ; \hat{r} = \frac{\vec{r}}{r}$$

we can calculate this for few examples

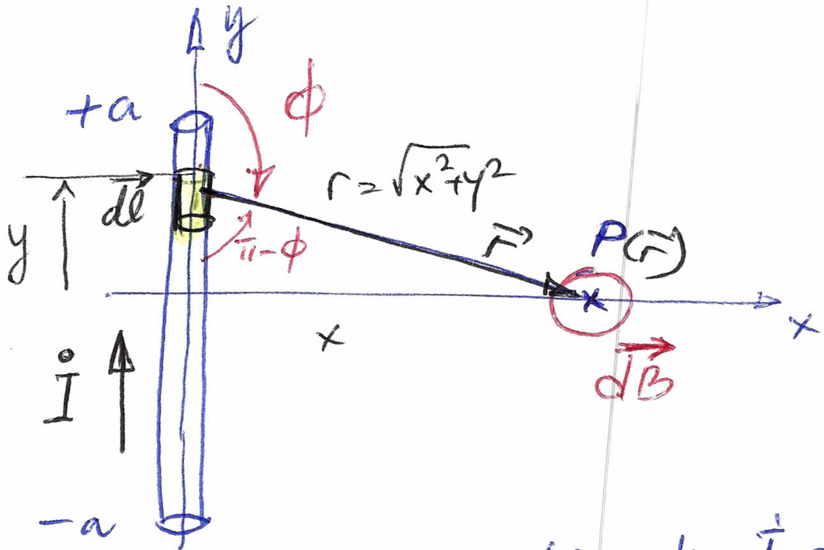
Magnetic field lines

The field vectors $d\vec{B}$ and the magnetic field lines of a current element are exactly like those of a positive charge dQ moving in the direction of the drift velocity $\vec{v}_d$

(3) Magnetic field of a stright current carrying conductor

We can use the law of Biot-Savart to calculate this.

This result is useful because stright conducting wires are found in all electric and electronic devices. We consider a conductor with the length $2a$ carrying a current I.



$$dl = dy$$

$$\sin\phi = \sin(\pi-\phi) = \frac{x}{\sqrt{x^2+y^2}}$$

$$dB = \frac{\mu_0}{4\pi}\, I\, \frac{dy\, \cancel{r} \sin\phi}{r^{\cancel{2}}}$$

$$\Rightarrow B = \frac{\mu_0 I}{4\pi}\int_{-a}^{a} \frac{x\,dy}{(x^2+y^2)^{3/2}}$$ after calculation

$$\Rightarrow \boxed{B = \frac{\mu_0 I}{4\pi}\,\frac{2a}{x\sqrt{x^2+a^2}}}$$

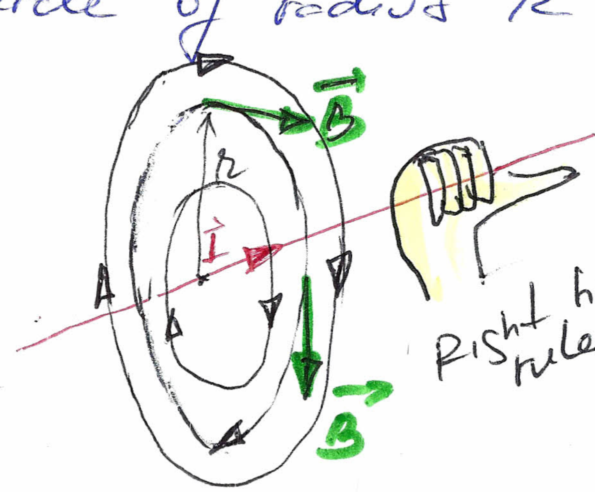
when $2a \gg x$ (infinitely long wire)
we can compute the limit $a \to \infty$
$\sqrt{x^2+a^2} \simeq a$ (neglect x with respect to a)

$$\Rightarrow B = \frac{\mu_0 I}{2\pi x}$$

This situation has axial symmetry about the y axis $\Rightarrow$
for all points on a circle of radius r
we will have:

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$

Magnetic field lines encircle the current that acts as their source.

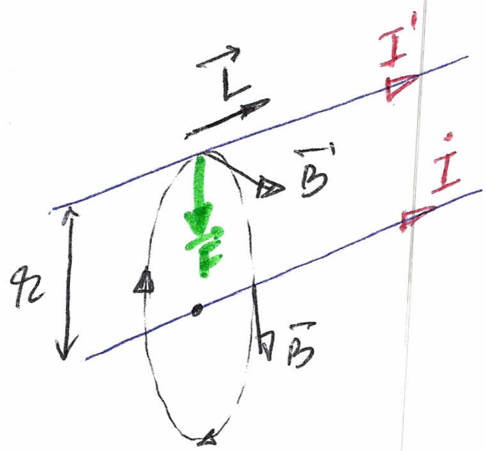


Right hand rule.

4) Force between parallel conductors

This force is important in situations where current carrying conductors are closed together.

We consider two straight conductors at distance R carrying currents I and I' in the same direction. Each conductor lies in the magnetic field set up by the other, so each experiences a force.



The lower conductor produces a field $\vec{B}'$ which at the position of the upper conductor is:

$$B = \frac{\mu_0 I}{2\pi R}$$

The force that this field exerts on a length L of the upper conductor is

$$\vec{F} = I'\vec{L} \times \vec{B}$$ where $\vec{L}$ is in the direction of I'

since $\vec{B} \perp \vec{L}$ =>

$$F = I'LB = \frac{\mu_0 I I' L}{2\pi R}$$ and thus, the force/unit length is

$$\boxed{\frac{F}{L} = \mu_0 \frac{I I'}{2\pi R}}$$

two, long, parallel current carrying conductors.

Applying the right-hand rule for

$$\vec{F} = I' \vec{L} \times \vec{B}$$ ⇒ force on upper conductor is oriented downward

⇒ attraction

Conclusion

Two parallel conductors carrying currents in same direction ATTRACT each OTHER.

Parallel conductors carrying currents in opposite direction repel each-other.

Magnetic force and defining Ampere

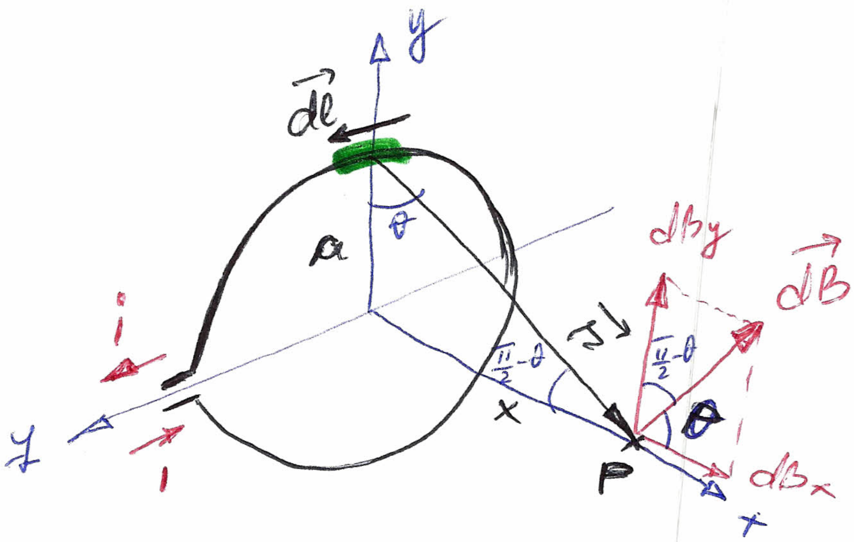
Attraction or repulsion between two straight parallel current carrying conductors is the basis of the official deffinition of the AMPÈRE.

One ampère is that unvarying current that, if present in two infinite length parallel conductors at 1 meter distance one with respect to the other in empty space causes each conductor to experience a force of exactly $2 \cdot 10^{-7}$ Newtons per meter of length.

5 Magnetic field of a circular current loop

We consider a circular conductor of radius a carrying a current I. We use Biot-Savart (fig) to calculate $\vec{B}$ at a distance x from the center.

From fig ⇒ $\vec{dl} \perp \vec{r}$ and $d\vec{B}$ lies in the xy plane.



$$r^2 = x^2 + a^2$$

$$dB = \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+a^2}$$ whose x and y components are.

$$dB_x = dB\cos\theta = \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+a^2}\frac{a}{\sqrt{x^2+a^2}}$$

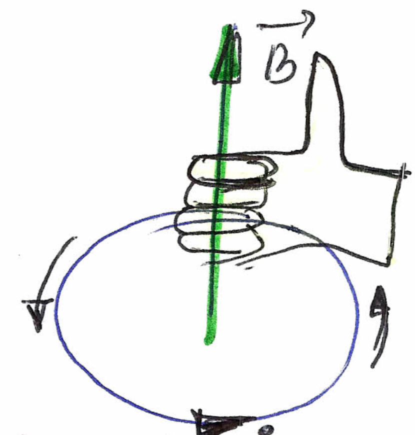
$$dB_y = dB\sin\theta = \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+a^2}\frac{x}{\sqrt{x^2+a^2}}$$

Due to symmetry considerations, the total field will have only ~~y~~ x component, (the y component cancels for two diametrally symmetric elements)

$$\Rightarrow \quad B_x = \int \frac{\mu_0 I}{4\pi}\frac{a\,dl}{(x^2+a^2)^{3/2}} = \frac{\mu_0 I a}{4\pi(x^2+a^2)^{3/2}}\int dl$$

$$\int dl = 2\pi a$$

$$\Rightarrow \boxed{B_x = \frac{\mu_0 I a^2}{2(x^2+a^2)^{3/2}}}$$

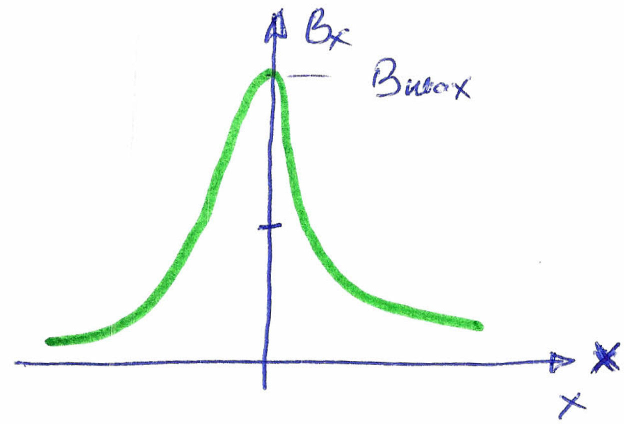


Right-hand rule

Magnetic field on the axis of a coil

Consider a coil with N identical loops (with some radius). The loops are closelly spaced so that all have same distance x with respect to point P. Then, the total field is N times the field of one loop

$$B_x = \frac{\mu_0 N I a^2}{2(x^2+a^2)^{3/2}}$$



x = 0 (center of the loop)

$$\boxed{(B_x)_{max} = \frac{\mu_0 N i}{2a}}$$

As we go out along the axis the field decreases in magnitude.

6 Ampère's law

So far we calculated $\vec{B}$ due to a current by integration of elementary $d\vec{B}$ due to a current element this approach is strongly equivalent to electric field calculation by integration.

For electric-field problem we saw that for situation with highly symmetric charge distributions it was often easier to use Gauss's law to find $\vec{E}$. There is likewise a law that allows us to calculate magnetic fields from highly symmetric current distributions. This is the law of Ampere.

Because the Gauss's law for $\vec{B}$ leads to

$\oint\!\!\oint \vec{B}\cdot d\vec{A} = 0$ for any closed surface, whether or not there are currents within the surface, this law cann'ot be used to calculate $\vec{B}$ produced by a particular current distribution.

Ampère law is formulated not in terms of magnetic flux, but rather in terms of line integral of $\vec{B}$ around a closed path, denoted by

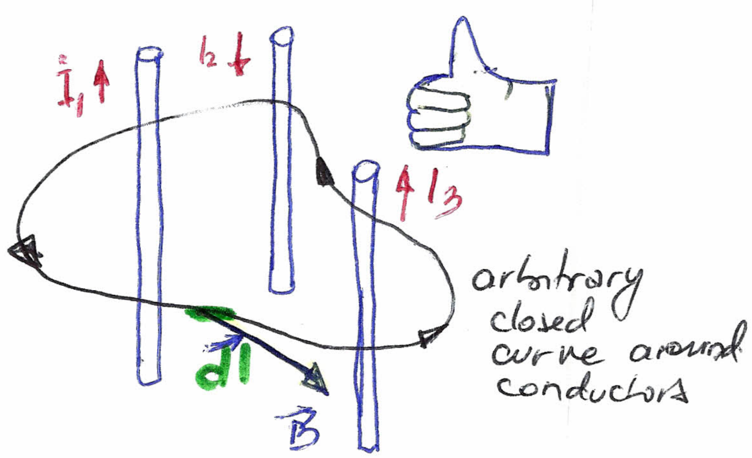
$$\oint \vec{B}\cdot d\vec{l}$$

(likewise integrals were used to calculate work or electric potentials)

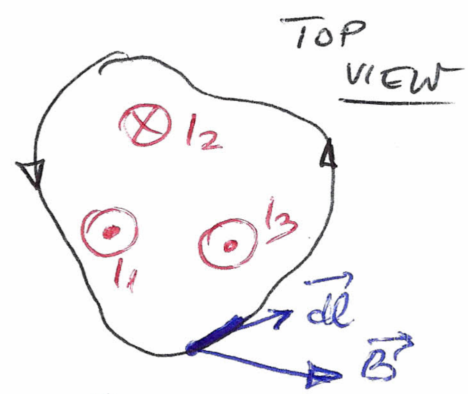
Ampère's law : general statement

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}}$$

I_{encl} = algebraic sum of all the currents enclosed or linked by the integration path



Curl the fingers of your right hand around the integration path. Your thumb points in the direction of positive current

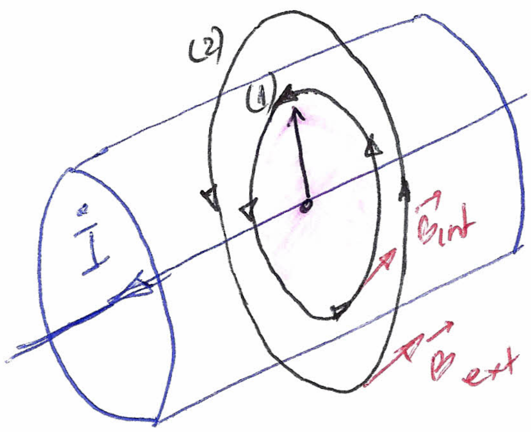


$I_{encl} = I_1 - I_2 + I_3$

(7) Applications of Ampère's law

Field of a long, straight, current-carrying conductor

- situation with cylindrical symmetry, so the integration path in the Ampère's law is a circle with radius r centered on conductor and having radius r



Consider that the conductor has the radius R

We consider 2 paths, corresp. to situations $r < R$ and $r > R$

(1) $r < R$ — the current circulating through the "pink" surface is:

$$I_{encl} = J \cdot \pi r^2 = \frac{I}{\cancel{\pi} R^2} \cancel{\pi} r^2 = \frac{I r^2}{R^2}$$

Ampère's law:

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{encl}; \quad B \oint dl = \mu_0 I_{encl}$$

$$B \, 2\pi r = \mu_0 I_{encl} \Rightarrow$$

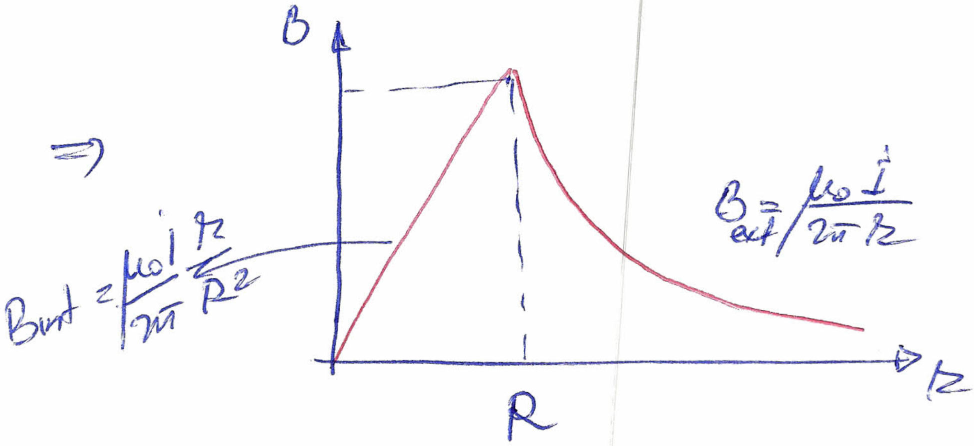
$$B \, 2\pi \cancel{r} = \mu_0 \frac{I r^{\cancel{2}}}{R^2} \Rightarrow \boxed{B_{int} = \frac{\mu_0 I}{2\pi} \frac{r}{R^2}}$$

(2) $r > R$ — $I_{encl} = I$

Ampère's law $\oint \vec{B} \cdot \vec{dl} = B \cdot 2\pi r = \mu_0 I \Rightarrow$

$$\boxed{B_{ext} = \frac{\mu_0 I}{2\pi r}}$$

$\Rightarrow$



Field of a solenoid

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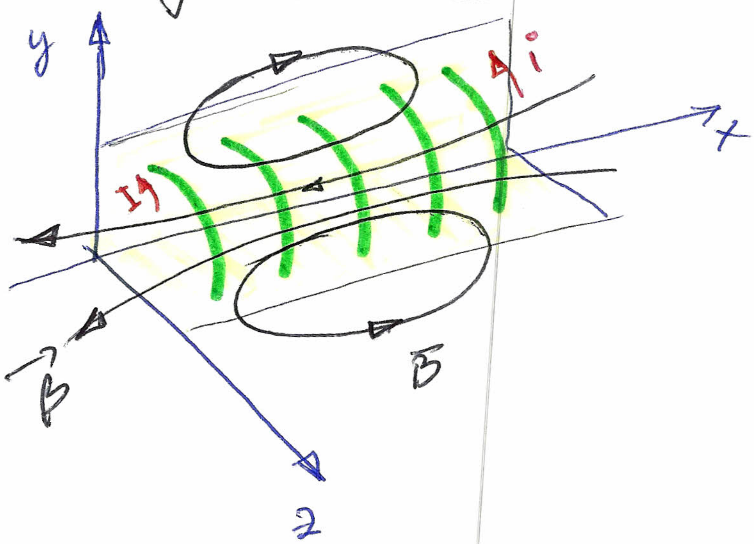
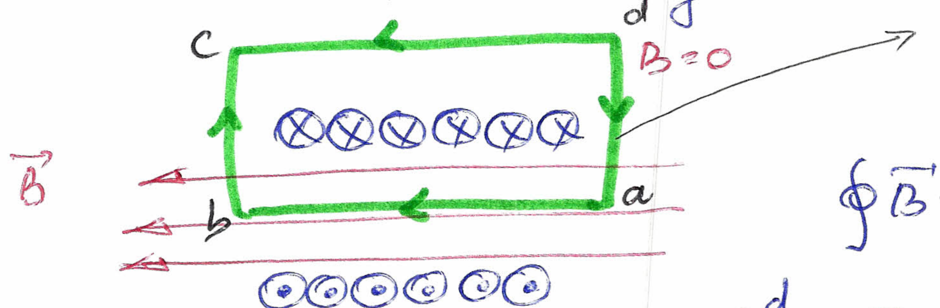


We assume that $\vec{B}$ is uniform inside the solenoid and zero outside.

In cross-section image:



$$\oint \vec{B}\cdot d\vec{l} = \int_a^b \vec{B}\, d\vec{l} + \underbrace{\int_b^c \vec{B}\, d\vec{l}}_{0\,(\vec{B}\perp d\vec{l})} + \underbrace{\int_c^d \vec{B}\cdot d\vec{l}}_{0\,(\vec{B}=0)} + \underbrace{\int_d^a \vec{B}\cdot d\vec{l}}_{0\,(\vec{B}\perp d\vec{l})} =$$

$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = \int_a^b B\, dl = BL$$

$$I_{encl} = NI = nLI$$

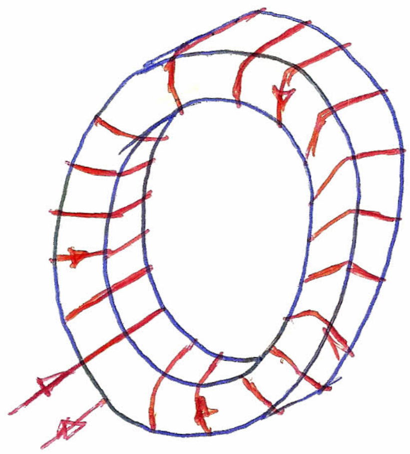
Ampère's law $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$

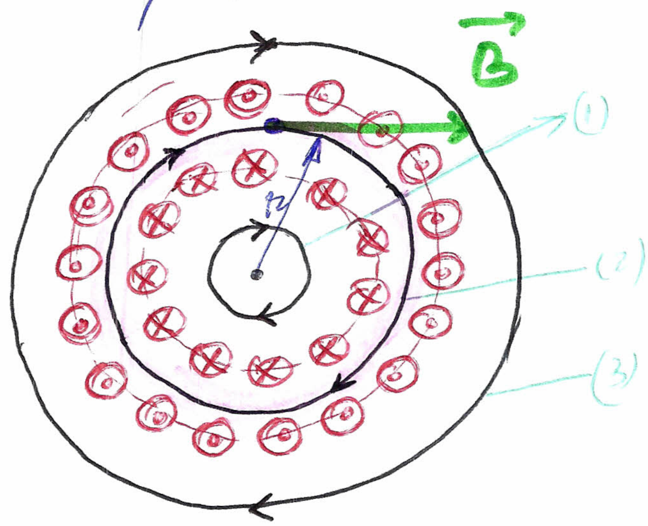
$\Rightarrow$ $\cancel{B}\cancel{L} = \mu_0 n \cancel{L} i$

$$\boxed{B = \mu_0 n I}$$

Field of a toroidal solenoid

Consider a doughnut-shaped toroidal solenoid tightly wound with N turns of wire carrying a current I. Find the magnetic field at all points.





Consider (3) paths

① Path (1) $\quad I_{encl} = 0 \quad \Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$

$\Rightarrow \vec{B} = 0$ everywhere on this path.

② Path (3) $\quad I_{encl} = I_{(\odot)} + I_{\otimes} = 0$

$\Rightarrow \vec{B} = 0$ everywhere.

③ Path (2) $\quad \oint \vec{B} \, d\vec{l} = 2\pi r B \qquad I_{encl} = NI$

$$\Rightarrow \boxed{B = \frac{\mu_0 N i}{2\pi r}} \quad \text{toroidal solenoid}$$

Ob : The magnetic field is confined in the area enclosed by the windings (in "pink").